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# Rate equations for a three-level system interacting with radiation

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It is shown that for short intervals the instantaneous population difference ($\Delta n$) varies linearly with time. The general solution of the rate equations is also obtained by taking constant induced transition probability between the lasing levels. The power emitted is found to vary exponentially with time which reaches a maximum value at a time ($t_m$) determined by relaxation times, induced transition probabilities and the spectral scheme.

## 1. INTRODUCTION

The instantaneous power output from an atomic system having a pair of energy levels in an inverted state under the action of an inducing field is given by

$$P(t) = \Delta n W_{12} h\nu_{12}, \qquad \dots \quad (1)$$

where $\Delta n$ is the instantaneous population difference and $W_{12}$ is the instantaneous transition probability.

We consider an assembly of non-interacting elementary system with three spectral levels. This model approximates to atoms with an isolated triplet of energy levels.

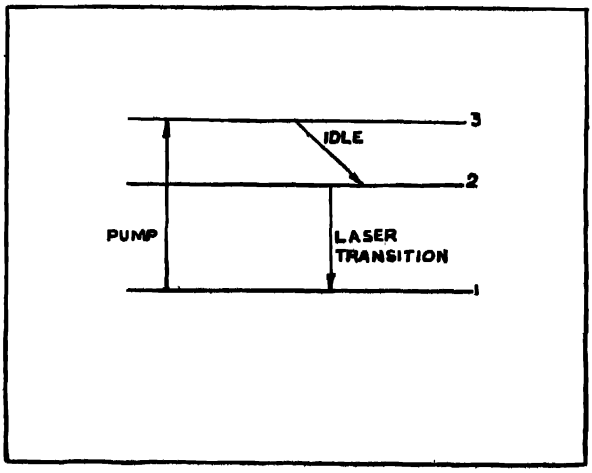


Fig. 1. Three level laser transition.

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If $n_1(t)$, $n_2(t)$ and $n_3(t)$ represent number of the particles in states labelled by 1, 2, 3 corresponding to energies $E_1, E_2, E_3\,(E_1 < E_2 < E_3)$.

The rate equations are

$$\frac{dn_1}{dt} = -p_{12}n_1+p_{21}n_2-p_{13}n_1+p_{31}n_3+W_{12}(n_2-n_1)+W_{13}(n_3-n_1), \quad \ldots \quad (2a)$$

$$\frac{dn_2}{dt} = -p_{21}n_2+p_{12}n_1-p_{23}n_2+p_{32}n_3+W_{12}(n_1-n_2), \quad \ldots \quad (2b)$$

$$\frac{dn_3}{dt} = -p_{31}n_3+p_{13}n_1-p_{32}n_3+p_{23}n_2+W_{13}(n_1-n_3), \quad \ldots \quad (2c)$$

where

$$n_1(t)+n_2(t)+n_3(t) = N, \quad \ldots \quad (3a)$$

$$n_i(t=0) = a_i, \quad \ldots \quad (3b)$$

$p_{ij} = \left(\frac{1}{\tau_{ij}}\right)$ and $W_{ij}$ are respectively the thermal and the stimulated transition probabilities per unit time between the levels $i$ and $j$.

At thermal equilibrium

$$\frac{p_{ij}}{p_{ji}} = \frac{n_j^e}{n_i^e} = \exp(-(E_j-E_i)/kT) \approx 1-\frac{h\nu_{ij}}{kT} \quad \ldots \quad (4)$$

$$= \epsilon_k, \text{ say},$$

$i, j, k = 1, 2, 3.$

Population inversion between the levels 1 and 2 takes place due to the action of a pumping field at frequency $\nu_{13}$ and the relaxation processes between the levels 3, and 2. The rate of growth of population inversion between the levels 1 and 2 depends mainly upon, (i) rate of pumping, (ii) initial equilibrium population of the levels and (iii) the rate of relaxation mechanism.

## 2. Solution of the Problem

(A) Here we solve for the instantaneous population inversion ($\Delta n$) between the lasing levels taking $W_{12}$ as a function of photon density ($n$) and hence time. In analogy with a two-level system with no loss we take $W_{12} = Rn$, where $n\alpha \sin^2 \alpha t$, $R$ is a constant and $\alpha = \mu_{12}E_0/2h$, $\mu_{12}$ being the matrix element for the component of the dipole moment along the field $\boldsymbol{E}$. In fact, $n\alpha \sin^2 \alpha t$ is the solution of the differential equation for photon number per atom (Venkatesh & Dixit 1971, Venkatesh & Roy 1971). Now from eqs. (2a) and (2b), we have

$$\frac{d^2\Delta n}{dt^2}+[L+2R\sin^2 \alpha t]\frac{d\Delta n}{dt}+[M+2RK\sin^2 \alpha t+4R\sin \alpha t.\cos \alpha t\;\alpha]\Delta n = F. \quad (5)$$

The population difference $(n_2-n_1)$ is then given by

$$\Delta n = a_0\left[1-\frac{M}{2}t^2-\left\{R\frac{(\mu_{12}E_0)^2}{6\hbar^2}-\frac{LM}{6}\right\}t^3+\dots\right]$$

$$+a_1t\left[1-\frac{L}{2}t+\left\{\frac{L^2}{6}-\frac{M}{6}\right\}t^2+\dots\right]$$

$$+\frac{F}{2}t^2\left[1-\frac{L}{3}t+\left\{\frac{L^2}{12}-\frac{M}{12}\right\}t^2+\dots\right], \quad \dots (6)$$

where

$$M = (p_{21}+p_{23}+p_{32})(p_{12}+p_{13}+p_{31}+2W_{13})-(p_{12}-p_{32})(p_{21}-p_{31}-W_{13})$$

$$L = p_{21}+p_{12}+p_{31}+p_{13}+p_{23}+p_{32}+2W_{13}$$

$$F = N[p_{12}p_{31}+p_{12}p_{32}+p_{32}p_{13}-p_{32}p_{21}-p_{31}p_{21}-p_{31}p_{23}$$
$$+W_{13}(p_{12}-p_{21}+p_{32}-p_{23})]. \quad \dots (7)$$

For small value of time $t$ we find that $\Delta n = a_0+a_1t$. Thus the linnear dependence of the population inversion is valid for short time only.

(B) Let us now consider $W_{12}$ as time independent. We solve for the instantaneous values of the populations from eq. (2) to get

$$n_i(t) = A_i+B_ie^{\alpha t}+C_ie^{\beta t}, \quad i = 1, 2, 3. \quad \dots (8)$$

where $A_i$, $B_i$, $C_i$, $\alpha$ and $\beta$ are constants (given in Appendix) involving $p_{ij}$'s and $W_{ij}$'s.

## 3. Power Emitted

Under condition when the spontaneous transition probability $p_{ij}$ may be considered much smaller than the induced transition probability $W_{ij}$, the output power (eq. (1)) at any time $t$ becomes

$$P(t) = \Delta n(t)W_{12}h\nu_{12}$$

$$=\frac{N}{3}[p_{21}(\epsilon_3-1)-p_{32}(\epsilon_1-1)]h\nu_{12}+\left\{\left[\frac{n_2^e-n_1^e}{2}\right]\left[1-\frac{W_{12}+W_{13}}{\sqrt{W_{12}^2+W_{13}^2-W_{12}W_{13}}}\right]\right.$$

$$-\frac{N}{6W_{12}}[p_{21}(\epsilon_3-1)-p_{32}(\epsilon_1-1)]\left[1+\frac{W_{12}+W_{13}}{\sqrt{W_{12}^2+W_{13}^2-W_{12}W_{13}}}\right]$$

$$\left.-\frac{[N-3n_2^e]W_{13}}{2\sqrt{W_{12}^2+W_{13}^2-W_{12}W_{13}}}\right\}W_{12}h\nu_{12}$$

$$\exp(-[W_{12}+W_{13}-\sqrt{W_{12}^2+W_{13}^2-W_{12}W_{13}}]t)$$

$$+\left\{\left[\frac{n_2^e-n_1^e}{2}\right]\left[1+\frac{W_{12}+W_{13}}{\sqrt{W_{12}^2+W_{13}^2-W_{12}W_{13}}}\right]\right.$$

$$-\frac{N}{6W_{12}}[p_{21}(\epsilon_3-1)-p_{32}(\epsilon_1-1]\left[1-\frac{W_{12}+W_{13}}{\sqrt{W_{12}^2+W_{13}^2-W_{12}W_{13}}}\right]$$

$$\left.+\frac{[N-3n_2^e]W_{13}}{2\sqrt{W_{12}^2+W_{13}^2-W_{12}W_{13}}}\right\}W_{12}h\nu_{12}$$

$$\times\exp(-[W_{12}+W_{13}+\sqrt{W_{12}^2+W_{13}^2-W_{12}W_{13}}]t). \qquad \ldots (9)$$

At time $t = 0$, the output power is $P(t = 0) = [n_2^e-n_1^e]W_{12}h\nu_{12}$ and is a negative quantity, since $n_2^e < n_1^e$. For large value of $t$, the output power is

$$P(t = \text{large}) = \frac{N}{3}[p_{21}(\epsilon_3-{}_1)-p_{32}(\epsilon_1-1)]h\nu_{12} = \text{const.} \qquad \ldots (10)$$

Thus we see that the power emitted is found to vary with time which ultimately attains a constant value.

From eq. (1) it is found that the output power will be maximum at $t = t_m$ where

$$t_m = \frac{1}{0.4343\,D}\log\left[-\frac{(C_2-C_1)}{(B_2-B_1)}\cdot\frac{(h+D)}{(h-D)}\right], \qquad \ldots (11)$$

and $h$ and $D$ are also constants depending on $p_{ij}$ and $W_{ij}$.

For the microwave frequency range in steady condition (Bloembergen 1956). the power output (eq. 10) reduces to

$$P = \frac{Nh^2\nu_{12}}{3KT}[p_{32}\nu_{32}-p_{21}\nu_{21}]. \qquad \ldots (12)$$


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## Appendix

$$A_i = -l_i/g,$$

$$B_i = \frac{a_i}{2} + \frac{l_i}{2g} + \frac{1}{D}\left[-k_i + \frac{hl_i}{2g} - \frac{a_i h}{2}\right],$$

$$C_i = \frac{a_i}{2} + \frac{l_i}{2g} - \frac{1}{D}\left[-k_i + \frac{hl_i}{2g} - \frac{a_i h}{2}\right],$$

$$\alpha = -[h - \sqrt{h^2 - 4g}]/2,$$

$$\beta = -[h + \sqrt{h^2 - 4g}]/2,$$

where

$$K_1 = -[(p_{21} + W_{12})(a_1 + a_2) + (p_{31} + W_{13})(a_1 + a_3) + a_1(p_{23} + p_{32})],$$

$$K_2 = -[(p_{12} + W_{12})(a_1 + a_2) + 2a_2 W_{13} + p_{32}(a_2 + a_3) + a_2(p_{31} + p_{13})],$$

$$K_3 = -[(p_{13} + W_{13})(a_1 + a_3) + p_{23}(a_2 + a_3) + a_3(p_{12} + p_{21} + 2W_{12})],$$

$$l_1 = -N[(p_{21} + W_{12})(p_{31} + p_{32} + W_{13}) + (p_{31} + W_{13})p_{23}],$$

$$l_2 = -N[(p_{12} + W_{12})(p_{31} + p_{32} + W_{13}) + (p_{13} + W_{13})p_{32}],$$

$$l_3 = -N[(p_{13} + W_{13})(p_{21} + p_{23} + W_{12}) + (p_{12} + W_{12})p_{23}],$$

$$h = [p_{12} + p_{21} + p_{13} + p_{31} + p_{23} + p_{32} + 2W_{12} + 2W_{13},$$

$$g = [(p_{21} + W_{12})(p_{31} + p_{13} + p_{32} + 2W_{13}) + (p_{12} + W_{12})(p_{31} + p_{23} + p_{32} + 2W_{13}) + (p_{31} + p_{13} + 2W_{13})p_{23} + p_{32}(W_{13} + p_{13})],$$

and

$$D = \sqrt{h^2 - 4g}.$$

## References


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